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de Sitter entropy from conformal field theory

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Abstract: We propose that the entropy of de Sitter space can be identified with the mutual entropy of a dual conformal field theory. We argue that unitary time evolution in de Sitter space restricts the total number of excited degrees of freedom to be bounded by the de Sitter entropy, and we give a CFT interpretation of this restriction. We also clarify issues arising from the fact that both de Sitter and anti de Sitter have dual descriptions in terms of conformal field theory.

Keywords: Superstring Vacua, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Thanks to AdS/CFT and M(atrix) theory, we have reasonable non-perturbative formulations of quantum gravity in backgrounds with negative or vanishing cosmological constant [1, 2, 3]. A natural next step is to study backgrounds with positive cosmological constant. However, despite considerable effort, our understanding of de Sitter space is quite limited. In part this is due to the difficulty of finding de Sitter solutions to string theory. But in part this is also due to the fact that a number of conceptual issues remain to be understood. Perhaps the foremost conceptual issue is to understand the entropy of de Sitter space.

Our study of de Sitter space will be based on the dS/CFT correspondence [4]; for other relevant work see [5]. We will propose an understanding of de Sitter entropy in this context. We are motivated by the idea that de Sitter entropy can be understood as entropy of entanglement evaluated on Cauchy surfaces in the bulk of de Sitter space. In the dual CFT, we will argue that de Sitter entropy can be understood as "mutual entropy" — a sort of euclidean entropy of entanglement. We will also point out some connections between physics in de Sitter and AdS spaces. It is natural to expect some connections, since the de Sitter metric can be obtained by an analytic continuation from AdS. Indeed this continuation may help to explain why both de Sitter and AdS are dual to CFT's.

An outline of this paper is as follows. In section 2 we review some basic properties of de Sitter space and the dS/CFT correspondence. In section 3 we present our interpretation

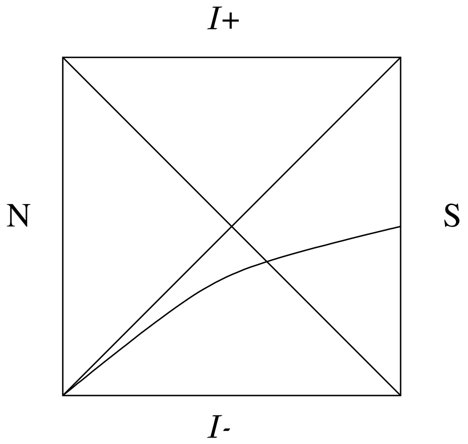

Figure 1: Penrose diagram for de Sitter space. We've indicated the two cosmological horizons as well as a fixed-time hypersurface in the planar coordinate system.

of de Sitter entropy in terms of mutual entropy in the CFT. In section 4 we discuss the analytic continuation from AdS to de Sitter space, and use this to gain insight into the dS/CFT correspondence. In section 5 we present an alternate interpretation of the entropy, based on the continuation from AdS, which is appropriate to static coordinates. In section 6 we discuss the entropy bounds arising from unitary time evolution.

2. de Sitter/CFT duality

We begin by recalling some properties of de Sitter space. $(d+1)$-dimensional de Sitter space dS_{d+1} is the hyperboloid

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 = \ell^2 \tag{2.1}$$

inside $\mathbb{R}^{d+1,1}$, where ℓ is the de Sitter radius. dS_{d+1} inherits its isometry group $\mathrm{SO}(d+1,1)$ from this embedding. This group is also the conformal group in d euclidean dimensions, which motivates the dS/CFT conjecture: the observables of quantum gravity in de Sitter space can be obtained from a euclidean conformal field theory in one less dimension. The CFT may well be non-unitary [4], although as we shall see it seems to have many of the properties of a unitary CFT.

The Penrose diagram for de Sitter space is shown in figure 1. A comoving observer can only interact with a finite region of the spacetime. For an observer at the south pole, this region is the right triangle in figure 1. This triangle can be described using the static metric

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \,. \tag{2.2}$$

The observer is located at $r = 0$, while the horizon is the sphere at $r = \ell$. The horizon has a temperature $T = 1/2\pi\ell$, while the entropy is given by the standard expression [6]

$$S = \frac{\text{area}}{4G} = \frac{\mathrm{vol}(S^{d-1})\ \ell^{d-1}}{4G} \,. \tag{2.3}$$

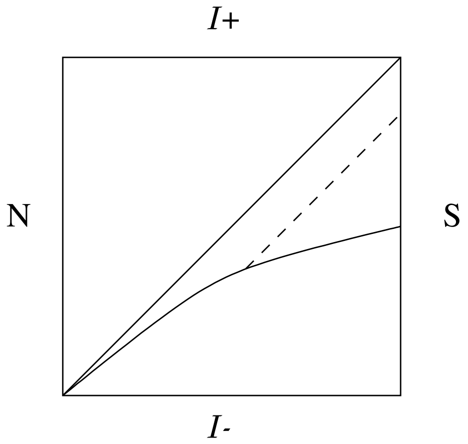

Figure 2: The dashed line is a contracting light-sheet, which begins at the south pole and intersects the constant-η hypersurface at $x^i x^i = R^2$.

We will also use planar coordinates, which cover the bottom and right triangles in figure 1. The metric is

$$ds^2 = \frac{\ell^2}{\eta^2}\left(-d\eta^2 + dx^i dx^i\right). \tag{2.4}$$

These coordinates parameterize the de Sitter hyperboloid (2.1) by setting

$$\begin{aligned} X^0 &= \frac{\eta}{2} - \frac{\ell^2}{2\eta} - \frac{1}{2\eta}x^i x^i \\ X^i &= \ell x^i/\eta \\ X^{d+1} &= \frac{\eta}{2} + \frac{\ell^2}{2\eta} - \frac{1}{2\eta}x^i x^i . \end{aligned} \tag{2.5}$$

The observer is located at $x^i = 0$, while the past horizon is located at $X^0 + X^{d+1} = 0$, or equivalently at

$$x^i x^i = \eta^2 . \tag{2.6}$$

That is, in the planar coordinate system, the cosmological horizon is a sphere of radius η.

We now discuss entropy bounds in planar coordinates. At time η, consider a spatial ball of arbitrary radius $x^i x^i \leq R^2$. Following Bousso's formulation of the holographic principle [7], the entropy contained in this ball is bounded by the entropy on the contracting light-sheet shown in figure 2, which in turn is bounded by

$$S = \frac{\text{surface area of ball}}{4G} = \frac{\text{vol}(S^{d-1})}{4G}\left(\frac{\ell R}{\eta}\right)^{d-1} . \tag{2.7}$$

When $R = \eta$, this reproduces the entropy of de Sitter space (2.3).

Planar coordinates are particularly convenient for discussing dS/CFT duality [4]. We take the continuum euclidean CFT to live at past infinity ($\eta = 0$), on the space parameterized by x^i. We identify the de Sitter isometry group with the group of conformal transformations of the CFT. For example, the de Sitter metric is invariant under

$$\eta \to \lambda\eta \qquad x^i \to \lambda x^i . \tag{2.8}$$

This isometry of de Sitter space corresponds to a dilation of the CFT, $x^i \to \lambda x^i$. The de Sitter invariant vacuum state of the gravity theory should correspond to the $\mathrm{SO}(d+1,1)$-invariant ground state of the CFT.

The de Sitter time coordinate η arises holographically, and is not manifest in the CFT. To understand its significance, we use the UV/IR correspondence. This states that an object at time η in de Sitter space corresponds to an excitation of scale size

$$|\delta x| = \eta \tag{2.9}$$

in the CFT. This is the de Sitter analog of the UV/IR relation familiar in AdS/CFT duality [8, 9].

Since objects on a constant-time hypersurface have a finite scale size in the CFT, we can encode them in a CFT with a short-distance cutoff. We will discuss the cutoff procedure in more detail in section 3, but the most straightforward procedure is to imagine that the CFT is put on a euclidean lattice with lattice spacing a. This regulated CFT contains all the degrees of freedom necessary to describe the hypersurface $\eta = a$ in de Sitter space.

3. de Sitter entropy in planar coordinates

From the point of view of bulk quantum gravity, it seems natural to associate an entropy with de Sitter space. This can be seen as follows. In the planar coordinate system (2.4), consider a fixed-time hypersurface $\eta =$ const. This is a Cauchy surface. But an observer located at the south pole can only interact with those degrees of freedom which are located on a part of this hypersurface, the part inside the cosmological horizon. Even if empty de Sitter space is associated with a pure state on this Cauchy surface, physics inside the horizon would have to be described in terms of a density matrix. It seems natural to identify the entropy of de Sitter space (2.3) with the entropy of this density matrix. That is, from the point of view of bulk quantum gravity, we would like to understand de Sitter entropy as arising from entropy of entanglement [10, 11].

More generally, the holographic bound (2.7) allows us to associate an entropy with a ball of arbitrary radius $x^i x^i \leq R^2$ on the hypersurface $\eta =$ const. This generalization will turn out to be very useful for us. The two entropies (2.3) and (2.7) are quite distinct and should not be confused. The cosmological entropy (2.3) is an intrinsic property of de Sitter space, arising from its global causal structure, and limits the number of degrees of freedom that any observer can interact with. The holographic bound (2.7), on the other hand, refers (for $R > \ell$) to degrees of freedom that are outside the cosmological horizon. An observer at the south pole will eventually be able to see these degrees of freedom, but can never influence them (see figure 2). The two notions of entropy coincide if one sets $R = \eta$, so that the surface of the ball coincides with the cosmological horizon.

We will assume that the holographic bound (2.7) is meaningful, and ask what it corresponds to in the CFT. The CFT lives on a space parameterized by x^i. Physics on the hypersurface $\eta =$ const. is described by a CFT with a short-distance cutoff $\delta x = \eta$. We will discuss the cutoff in more detail below, but for now imagine putting the CFT on a euclidean lattice with lattice spacing equal to η. We also assume that the CFT has a lagrangian description, so that it can be understood in terms of degrees of freedom $\phi_{i,n}$. Here i labels the various lattice sites, and n labels the degrees of freedom at each site. These degrees of freedom will fluctuate according to a probability distribution $P(\phi) = \frac{1}{Z} e^{-S(\phi)}$, where S is the euclidean action.[1]

We are interested in the region $x^i x^i \leq R^2$ of the hypersurface $\eta =$ const. It seems natural to expect that this region of the bulk spacetime is holographically encoded in a finite region of the cutoff CFT, namely a ball $x^i x^i \leq R^2$. Note that the region inside the cosmological horizon should be encoded in a single lattice site, see (2.6).

Our proposal is that the entropy (2.7) associated with regions of de Sitter space can be understood as arising from correlations between CFT degrees of freedom located on opposite sides of the sphere $x^i x^i = R^2$. These correlations can be quantified in terms of "mutual entropy," a concept which we will define more precisely momentarily. Mutual entropy can be thought of as a euclidean analog of entropy of entanglement. Stated more precisely, our proposal is that entropy of entanglement in the bulk of de Sitter space is dual (in the sense of the dS/CFT correspondence) to mutual entropy in the CFT.

This gives rise to a rather attractive picture of time evolution. From the bulk de Sitter point of view, as time evolves new degrees of freedom can flow in through the past cosmological horizon. Nonetheless the total entropy inside the horizon remains constant. From the CFT point of view the region inside the horizon corresponds to a single lattice site, and time evolution corresponds to a block-spin transformation. As time (RG parameter) evolves, the single lattice site corresponds to a larger and larger region of the CFT spacetime. Nonetheless the total entropy associated with the site remains constant. We also get an amusing picture of what an observer means in the euclidean CFT: an inertial observer in de Sitter space sits on a single lattice site of the CFT.

3.1 Mutual entropy

Consider a collection of random variables, which we separate into two sets of degrees of freedom X and Y. We will have in mind that we can measure X but are unable to observe Y. From the joint probability distribution $P(X,Y)$ we can construct probability distributions for X and Y.

$$\begin{aligned} P(X) &= \int dY\, P(X,Y) \\ P(Y) &= \int dX\, P(X,Y)\,. \end{aligned}$$

Even if we could observe both X and Y we would have some lack of *a priori* information due to the fact that X and Y fluctuate. This lack of information can be measured by the

[1] It is not clear whether this is the correct framework, since the CFT which is dual to de Sitter space might be non-unitary.

entropy[2]

$$H_{XY} = -\int dX dY \, P(X,Y) \log P(X,Y) \, .$$

Likewise we can associate an entropy

$$H_X = -\int dX \, P(X) \log P(X)$$

with the fluctuations in X, and an entropy

$$H_Y = -\int dY \, P(Y) \log P(Y)$$

with the fluctuations in Y. Now suppose that we can only measure X. Restricting to these degrees of freedom leads to an increase in the entropy, due to the fact that we are not able to observe correlations between X and Y. This increase can be quantified by defining the mutual entropy (also known as mutual information)

$$I_{XY} = H_X + H_Y - H_{XY} \, . \tag{3.1}$$

Note that mutual information, since it involves entropy differences, is free from the divergences mentioned in[2].

Mutual entropy is a familiar concept in information theory [12]. It satisfies a number of properties,

- $I_{XY} \geq 0$, with equality iff X and Y are statistically independent (that is, iff $P(X,Y) = P(X)P(Y)$).
- For discrete random variables

  $$I_{XY} \leq \min(H_X, H_Y) \tag{3.2}$$

  with equality iff X and Y are perfectly correlated (that is, iff the value of X uniquely determines the value of Y, and visa versa).

3.2 Mutual entropy in quantum field theory

In this section we study mutual entropy in the context of euclidean quantum field theory. Consider a quantum field which evolves from an initial field eigenstate $|q_i\rangle$ at time τ_i to a final eigenstate $|q_f\rangle$ at time τ_f. This is described by the configuration space path integral

$$Z = \langle q_f, \tau_f | q_i, \tau_i \rangle = \int\limits_{\substack{\phi(\tau_i)=q_i \\ \phi(\tau_f)=q_f}} \mathcal{D}\phi \, e^{-S} \, .$$

[2] If a random variable takes on discrete values, one can define entropy by $H = -\sum_k P_k \log P_k$. In the continuum limit $P_k \to P(x)dx$ and the entropy diverges. As usual we define the entropy of a continuous distribution by discarding this divergence.

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Cutting open the path integral at some intermediate time τ, one has

$$Z = \int dq\, \langle q_f, \tau_f | q, \tau\rangle \langle q, \tau | q_i, \tau_i\rangle$$

so the probability distribution for q is

$$P(q) = \frac{1}{Z}\langle q_f, \tau_f | q, \tau\rangle \langle q, \tau | q_i, \tau_i\rangle\,.$$

Now let us separate the field into

$$\begin{aligned} X &= \{\text{degrees of freedom between } \tau_i \text{ and } \tau\} \\ Y &= \{\text{degrees of freedom between } \tau \text{ and } \tau_f\}\,. \end{aligned}$$

In a continuum field theory it doesn't matter whether we regard q as belonging to X or Y. The joint probability distribution for these degrees of freedom is

$$P(X,Y) = \frac{1}{Z} e^{-S(X,Y)}$$

with an associated entropy

$$H_{XY} = -\int \mathcal{D}X\mathcal{D}Y\, P(X,Y) \log P(X,Y) = \log Z + \langle S(X,Y)\rangle\,.$$

The probability distribution for X is

$$P(X) = \int \mathcal{D}Y\, P(X,Y) = \frac{1}{Z} e^{-S(X)} \langle q, \tau | q_i, \tau_i\rangle\,,$$

where we have regarded q as belonging to the set X, and have made use of the locality axiom $S(X,Y) = S(X) + S(Y)$. The entropy associated with X is then

$$\begin{aligned} H_X &= -\int \mathcal{D}X\, P(X) \log P(X) \\ &= -\int dq\, P(q) \log\left(\frac{1}{Z}\langle q, \tau | q_i, \tau_i\rangle\right) + \int \mathcal{D}X\mathcal{D}Y\, P(X,Y) S(X)\,. \end{aligned}$$

Likewise the entropy for Y is

$$H_Y = -\int dq\, P(q) \log\left(\frac{1}{Z}\langle q_f, \tau_f | q, \tau\rangle\right) + \int \mathcal{D}X\mathcal{D}Y\, P(X,Y) S(Y)\,,$$

where now we're regarding q as belonging to the set Y. Putting these results together, the mutual entropy is given by

$$I_{XY} = -\int dq\, P(q) \log P(q)\,. \tag{3.3}$$

In this derivation we have implicitly used a configuration-space path integral, so we take the probability distribution appearing in (3.3) to be determined by the wavefunction of the system written in configuration space.

We will be most interested in the limit $\tau_i \to -\infty$ and $\tau_f \to +\infty$. Then $P(q) \to \psi_0^* \psi_0(q)$, where ψ_0 is the ground state wavefunction. That is, in this limit mutual entropy is associated with the ground state wavefunction of a system, and is obtained by regarding the wavefunction as providing a statistical distribution $P(q)$ for the possible configurations of the system at time τ.[3]

We are now in a position to compute the mutual entropy for a scalar field of mass m. For simplicity we work in two spacetime dimensions, on a cylinder $\mathbb{R}\times S^1$. Decomposing the field into Fourier modes around the S^1 yields an infinite collection of harmonic oscillators with frequencies $\omega_n = \sqrt{(n/R)^2 + m^2}$, where $n \in \mathbb{Z}$ and R is the radius of the spatial circle. The entropy associated with a harmonic oscillator in its ground state is $-\frac{1}{2}\log\omega$, so

$$I_{XY} = -\frac{1}{2}\sum_{n\in\mathbb{Z}} \log\sqrt{\left(\frac{n}{R}\right)^2 + m^2}\,.$$

The sum is divergent. Subtracting the contribution of a Pauli-Villars regulator field with mass $M \to \infty$ we have

$$I_{XY} = -\frac{1}{4}\sum_{n\in\mathbb{Z}} \log\frac{n^2 + m^2R^2}{n^2 + M^2R^2} = -\frac{1}{2}\log\frac{\sinh \pi m R}{\sinh \pi M R}\,. \tag{3.4}$$

3.3 Mutual entropy in conformal field theory

We now turn to the study of mutual entropy in conformal field theory, and present a heuristic argument that mutual entropy is proportional to the central charge of the CFT.

First let us count the degrees of freedom necessary to describe a cut-off CFT. One way to count the degrees of freedom with a given cutoff Λ is to study the system at a finite temperature $T \sim \Lambda$. The resulting thermal entropy counts the degrees of freedom with energies up to the cutoff. A two-dimensional unitary CFT at temperature T has an entropy per unit length given by [13]

$$\frac{S}{L} = \frac{\pi c T}{3}\,, \tag{3.5}$$

where c is the central charge. Since $L\Lambda$ is the number of spatial lattice sites we see that the number of degrees of freedom per lattice site needed to describe the cutoff theory is proportional to the central charge.

$$\frac{S}{L\Lambda} \sim \frac{\pi c}{3} \tag{3.6}$$

One can of course get the same result using the entropy-energy relationship. Let x be the number of degrees of freedom per lattice site. Then one has

$$\frac{S}{L\Lambda} \sim x\,, \qquad \frac{E}{L\Lambda} \sim x\Lambda \tag{3.7}$$

as well as the Cardy formula

$$S \sim \sqrt{cEL}\,. \tag{3.8}$$

Again one finds $x \sim c$.

[3] Mutual entropy should not be confused with the usual notion of entropy in quantum statistical mechanics, which vanishes in this case since ψ_0 is a pure state.

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Thus the conformal field theory can be described in terms of approximately c degrees of freedom per lattice site. We expect the mutual entropy to be bounded by the available number of degrees of freedom. If the bound is saturated, we expect the mutual entropy to be proportional to c. In particular the entropy associated with a single site in the lattice should be of order c.

As a concrete example of a conformally invariant theory, consider a massless scalar field. Using the result (3.4), the entropy in the massless limit is given by

$$I_{XY} = \frac{\pi MR}{2} - \frac{1}{2}\log(2\pi RM)\,,$$

where we have suppressed the contribution of the zero mode. In radial quantization, this is the mutual entropy of a $c = 1$ CFT associated with a disc of radius R, regulated with a Pauli-Villars cutoff.

3.4 Mutual entropy and de Sitter entropy

We now discuss the extent to which mutual entropy in a CFT has the right properties to account for the entropy of de Sitter space.

For simplicity we will concentrate on the case of dS_3/CFT_2. In this case the central charge is known [4],

$$c = \frac{3\ell}{2G}\,.$$

Thus the holographic entropy (2.7) can be written as

$$S = \frac{\pi c R}{3\eta}\,. \tag{3.9}$$

The salient features are that the entropy is proportional to the central charge, scales linearly with the radius R, and scales inversely with the time η.

As we discussed in section 3.4, it is reasonable to expect that the mutual entropy of a conformal field theory is proportional to the central charge. Moreover, as we saw in section 3.3, the mutual entropy of a two-dimensional field theory is linearly divergent. For example, for a scalar field with a Pauli-Villars cutoff the entropy is

$$I_{XY} = \frac{\pi MR}{2}$$

when $MR \gg 1$. This displays the desired linear scaling with R. The de Sitter CFT should have a short-distance cutoff at $\delta x = \eta$, or equivalently at $M \approx 1/\eta$. So the mutual entropy also has the desired inverse scaling with η.

Having mentioned these features, let us also point out some potential difficulties. One issue is that it is not clear whether the arguments of section 3.4, that the mutual entropy of a CFT is proportional to c, apply to non-unitary CFT's. Another issue is that to obtain the entropy associated with the cosmological horizon we need to set $R = \eta$. That is, we need to study the entropy associated with a region in the CFT whose size is set by the UV cutoff. The resulting entropy will be non-universal, since it depends on exactly how one regulates the CFT. A related issue arises because the entropy is linearly divergent. The coefficient of a linear divergence is non-universal, so it would seem that one could not hope to get the right coefficient (the $\pi/3$ in (3.9)).

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Let us mention how we think some of these difficulties are resolved. We believe the de Sitter / CFT correspondence picks out a preferred regulator for the CFT. Roughly speaking, a slice η = const. in de Sitter space can be described by a CFT with a short distance lattice cutoff at $\delta x = \eta$. However the precise cutoff procedure should be given by using a bulk-to-boundary propagator to map localized objects in the bulk to smeared-out excitations on the boundary. The size of the boundary excitations will be roughly $\delta x = \eta$. But the bulk-to-boundary propagator should give a precise smearing function, or equivalently a precise way of regulating the CFT. If one could calculate mutual entropy using this preferred regulator, one might be able to reproduce the correct coefficient in the entropy.

A similar issue of regulator dependence arises in AdS/CFT duality. In the AdS/CFT context holography states that the number of degrees of freedom of a cut-off CFT is equal to one quarter of the area of an appropriate surface in Planck units [8]. To get the one quarter a specific regularization scheme must be employed, but the precise scheme has not been explicitly worked out.

4. From AdS to de Sitter space

Since the metric of AdS can be analytically continued to give the metric of de Sitter space, and since the relationship between AdS and CFT is well-understood, one might hope to use AdS/CFT duality to learn more about dS/CFT. In this section we adopt this approach, and use it to gain some insight into the non-unitarity of the de Sitter CFT, as well as the origins of supersymmetry breaking in de Sitter space.

First let us explain what we mean by analytically continuing from AdS to de Sitter space. The duality between string theory in an AdS background and conformal field theory identifies the Hilbert space of the gravity theory with the Hilbert space of the CFT [14]. This identification implies the existence of a set of "bulk operators" in the CFT, which we denote $\mathcal{O}_{\mathrm{AdS}}(\tilde{t},\tilde{x},\tilde{r})$, where $(\tilde{t},\tilde{x})$ are the coordinates of the CFT and $\tilde{r}$ is the extra radial coordinate of the AdS gravity theory. These operators create states in the CFT which are dual to localized excitations in the bulk. As long as we are in a regime where supergravity is valid, these bulk operators suffice to describe quasi-local physics in the bulk of AdS.[4] We do not know how to construct a complete set of bulk operators, but all we will use here is the fact that they exist.

These bulk operators depend on an insertion point in the bulk of AdS, which is labeled by the coordinates of the CFT $\tilde{t}$, $\tilde{x}$ plus the extra radial coordinate $\tilde{r}$. The bulk operators can be expanded in terms of CFT operators, with coefficients depending on the CFT coordinates as well as on the radial direction.

$$\mathcal{O}_{\mathrm{AdS}}(\tilde{t},\tilde{x},\tilde{r}) = \sum_i \int dtdx\, f_i(t,x,\tilde{t},\tilde{x},\tilde{r}) O_i(t,x)\,. \tag{4.1}$$

Here the O_i are a basis of local operators in the CFT. Once one has these operators one can compute their correlation functions, which in principle encode all information about

[4]For a possible set of such bulk operators see [15].

the bulk. In particular given the correlation functions one ought to be able to recover the bulk metric. If we start with a lorentzian CFT on $\mathbb{R} \times S^{d-1}$, then we would obtain the metric of AdS_{d+1}, for instance in the form (with signature $- + + + \cdots$)

$$ds^2 = -\left(1+\frac{\tilde{r}^2}{\ell^2}\right)d\tilde{t}^2 + \left(1+\frac{\tilde{r}^2}{\ell^2}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 d\Omega_{d-1}^2 \,. \tag{4.2}$$

These coordinates cover all of global AdS provided $-\infty < \tilde{t} < \infty$. The AdS radius ℓ corresponds to a parameter in the CFT.

Suppose we take the bulk operators $\mathcal{O}_{\mathrm{AdS}}(\tilde{t}, \tilde{x}, \tilde{r})$ and analytically continue them to complex values of $\tilde{t}$ and $\tilde{r}$. This allows us to define

$$\mathcal{O}_{\mathrm{dS}}(t, \tilde{x}, r) \equiv \mathcal{O}_{\mathrm{AdS}}(it, \tilde{x}, ir) \,. \tag{4.3}$$

These double Wick rotated operators $\mathcal{O}_{\mathrm{dS}}(t, \tilde{x}, r)$ are still formally operators in the CFT (now a euclidean CFT), so in principle one can compute their correlation functions and recover the metric of the bulk spacetime. But the metric that one will deduce from this procedure is now

$$ds^2 = \left(1-\frac{r^2}{\ell^2}\right)dt^2 - \left(1-\frac{r^2}{\ell^2}\right)^{-1} dr^2 - r^2 d\Omega_{d-1}^2 \,. \tag{4.4}$$

This is the metric of lorentzian de Sitter space, obtained in the static coordinates (2.2), but with a flipped signature $(+ - - - \cdots)$. The metric has a coordinate singularity at $r = \ell$. But by extending the range of the radial coordinate to $0 < r < \infty$, we will regard these coordinates as covering one half of de Sitter space, which includes either past infinity or future infinity. The dual CFT then lives on the boundary $r \to \infty$, with topology $\mathbb{R} \times S^{d-1}$.

Similar analytic continuations can be set up in other coordinate systems. For example, one can start from euclidean AdS with metric

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + d\tau^2 + dx^i dx^i) \,. \tag{4.5}$$

Setting $\eta = iz$ gives

$$ds^2 = \frac{\ell^2}{\eta^2}(d\eta^2 - d\tau^2 - dx^i dx^i) \tag{4.6}$$

which is just de Sitter space in planar coordinates with signature $(+ - - - \cdots)$. These coordinates also cover half of de Sitter space, and the corresponding CFT lives on $\mathbb{R}^d$.

Let us make some comments on these analytic continuations. First of all, even if one starts from a unitary CFT, it is quite possible that the double Wick rotated operators are not in unitary representations of the conformal group (for instance they may create non-normalizable states). Also note that we are ignoring the fate of the internal compactification manifold, such as the S^5 of $AdS_5 \times S^5$. The internal manifold can perhaps be dealt with along the lines of [16].

In both examples we find that a euclidean CFT naturally describes half of de Sitter space. The boundary of AdS is mapped to the asymptotic past (or future) of de Sitter

($r \to \infty$ or $\eta \to 0$), so one can regard the CFT as living at past (or future) infinity. The observables of the gravity theory live on the boundary, and the description of the bulk of de Sitter is holographic.

To understand the continuation in more detail, consider a scalar field of mass m^2 in lorentzian anti de Sitter space, with equation of motion

$$(\Box_{\mathrm{AdS}} - m^2)\phi_{\mathrm{AdS}} = 0\,. \tag{4.7}$$

If we rotate $t = i\tilde{t}$, $r = i\tilde{r}$ and call the double Wick rotated field ϕ_{dS}, it satisfies

$$(\Box_{\mathrm{dS}} - m^2)\phi_{\mathrm{dS}} = 0 \tag{4.8}$$

but now with a flipped signature, which gives the scalar field a mass equal to $-m^2$. This suggests that a scalar field in de Sitter space with mass $+m^2$ corresponds to a scalar field in AdS with mass $-m^2$. This nicely explains the conformal dimensions of operators that one gets from dS/CFT duality. An operator dual to a field of mass m^2 in de Sitter space has a conformal dimension [4]

$$h_{\pm} = \frac{1}{2}(d \pm \sqrt{d^2 - 4m^2\ell^2})\,. \tag{4.9}$$

But this is the conformal dimension of an operator associated with a field of mass $-m^2$ in AdS.

The full correspondence between fields in de Sitter and AdS, however, involves more than just a double Wick rotation. To see this, consider starting in de Sitter space with a scalar field of mass m^2. The Wightman function for such a field is (in planar coordinates, in the so-called euclidean vacuum [17])

$$G_{\mathrm{dS}}(x,x') = F(h_+, h_-, \frac{d+1}{2}, \frac{1+P(x,x')}{2})\,, \tag{4.10}$$

where

$$\begin{aligned} h_{\pm} &= \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 - m^2\ell^2} \\ P(x,x') &= \frac{\eta^2 + \eta'^2 - |x-x'|^2}{2\eta\eta'}\,. \end{aligned}$$

When we analytically continue $z = -i\eta$ to obtain the euclidean AdS metric (4.5), G_{dS} does not turn into a Greens function for a scalar field of mass $-m^2$ in AdS [18, 19]. Instead note that the hypergeometric function satisfies

$$\begin{aligned} F(\alpha,\beta,\gamma,z) &= (1-z)^{-\alpha}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}F\left(\alpha,\gamma-\beta,\alpha-\beta+1,\frac{1}{1-z}\right) + \\ &\quad + (1-z)^{-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}F\left(\beta,\gamma-\alpha,\beta-\alpha+1,\frac{1}{1-z}\right)\,. \end{aligned} \tag{4.11}$$

Each term on the right hand side of this equation is proportional to the Greens function for a scalar field in euclidean AdS with mass squared equal to $-m^2$ [20]. That is, the de Sitter Greens function becomes a sum of AdS Greens functions.

What does this imply for the relation between de Sitter space and AdS? First consider a scalar field in de Sitter space with mass satisfying

$$\frac{d^2}{4} - 1 < m_{\rm dS}^2 \ell^2 < \frac{d^2}{4} \,. \tag{4.12}$$

The mass of the corresponding field in AdS satisfies

$$-\frac{d^2}{4} < m_{\rm AdS}^2 \ell^2 < 1 - \frac{d^2}{4} \,. \tag{4.13}$$

In this mass range a scalar field in AdS can be quantized in two inequivalent ways, corresponding to two possible choices of boundary conditions at infinity [21, 22]. The different boundary conditions give rise to distinct Greens functions, and in (4.11) *both* Greens functions appear. This means that a single field in de Sitter space must be associated with a pair of fields in AdS. In the dual AdS CFT this means we must have a pair of operators, one of dimension h_+ and one of dimension h_-.[5] Going back to de Sitter space, a single scalar field in de Sitter space should correspond to a bulk operator $\mathcal{O}_{\rm dS}$ in the de Sitter CFT. $\mathcal{O}_{\rm dS}$ can be constructed by taking a linear combination of the two corresponding bulk operators in the AdS CFT and performing a Wick rotation $z = -i\eta$.

Outside the mass range (4.12) the situation is more subtle. The identity (4.11) still holds, but there is only a single quantization of a scalar field in AdS, and one or both of the operators in the AdS CFT will violate the unitarity bound.

To summarize, each field in the mass range (4.12) in de Sitter space corresponds to a pair of fields in AdS. Having both fields in AdS breaks supersymmetry, since AdS supersymmetry requires particular boundary conditions on the fields [21, 23]. The de Sitter CFT bulk operators are linear combinations of the corresponding Wick rotated AdS CFT bulk operators.

5. Entropy in static coordinates

In this section we discuss the entropy of de Sitter space from the point of view of the CFT on $\mathbb{R} \times S^{d-1}$, i.e. in the static r, t coordinate system. First let us recall how one computes the entropy of de Sitter space. One takes a smooth euclidean section of de Sitter space, which is just the round sphere. In static coordinates this is done by rotating t to it. To get a smooth metric one has to periodically identify imaginary time with period $2\pi\ell$. One is left with a space parameterized by $0 < it < 2\pi\ell$, $0 < r < \ell$, $\Omega \in S^{d-1}$. Then one computes the euclidean action, which is minus the entropy (the energy is zero).

Analytically continuing this procedure to AdS it seems we are throwing out the $\tilde{r} > \ell$ part of AdS. The resulting entropy should be interpreted as entropy of entanglement between the two regions $\tilde{r} > \ell$ and $\tilde{r} < \ell$ in the AdS vacuum state. In the dual CFT the $\tilde{r}$ coordinate arises holographically, and is related to the cutoff on the CFT [8]. So from the CFT point of view this entropy does not arise from entanglement between two regions of space, but rather arises from entanglement between different energy scales.

[5] From the point of view of the AdS CFT, the mass range (4.13) is the unitarity bound for these two operators: h_+, h_- real and larger than $\frac{d}{2} - 1$ [22].

Entanglement entropy between different energy scales (unlike between two regions of space) vanishes for a free theory, since a free theory has a vacuum wavefunctional which is a tensor product in momentum space. But in an interacting theory the appearance of non-trivial higher-point correlation functions implies entanglement between energy scales. In order to actually compute this entropy from the CFT we would have to know the vacuum wavefunctional of the CFT. Lacking this, we use the fact that the CFT is equivalent to a semi-local field theory in the bulk of AdS. If the bulk theory was really local then the entanglement entropy between $\tilde{r} > \ell$ and $\tilde{r} < \ell$ would be UV divergent, and proportional to the area of the boundary between the two regions. This reflects the fact that entanglement entropy is measuring correlations between the two regions, given some cutoff scale. As the bulk theory is not really local one cannot take the UV cutoff to zero. A natural cutoff in AdS is provided by the Planck length, so one expects the entropy of entanglement to be proportional to

$$\frac{A}{G} = \frac{\mathrm{vol}(S^{d-1})\,\ell^{d-1}}{G}\,. \tag{5.1}$$

Without further information we cannot deduce the coefficient, but we can put a bound on it. The entanglement entropy between the two regions cannot exceed the log of the number of degrees of freedom residing in either one of the regions, since the entanglement entropy can be computed from a density matrix in either region. From AdS/CFT we know that gravity in the region $\tilde{r} < \ell$ is described by a finite number of degrees of freedom, while gravity in the region $\tilde{r} > \ell$ has an infinite number of degrees of freedom. This gives an upper bound on the entanglement entropy, namely the number of degrees of freedom residing at $\tilde{r} < l$, which is given by

$$\frac{\mathrm{vol}(S^{d-1})\,\ell^{d-1}}{4G}\,. \tag{5.2}$$

This is exactly the de Sitter entropy. Thus de Sitter entropy saturates the bound, and the entanglement entropy is maximal.

The degrees of freedom within one AdS radius can be described by a dimensionally reduced theory [24]. This is very similar to the picture we discussed in section 3, where the region inside the horizon of de Sitter space is described by a single lattice site. It would be very interesting to understand the relation between the two pictures of the entropy we have developed, for planar coordinates (dual to CFT on $\mathbb{R}^d$) and static coordinates (dual to CFT on $\mathbb{R} \times S^{d-1}$).

6. Entropy and time evolution

We have seen that the entropy of de Sitter space can be understood as entropy of entanglement between energy scales in the vacuum state of the CFT on $\mathbb{R} \times S^{d-1}$, or in terms of mutual entropy in the CFT on $\mathbb{R}^d$. We now discuss a dynamical role which the entropy plays in de Sitter space [25].

Unitary time evolution requires that two different initial states will evolve to two different final states. However since time evolution in de Sitter space is associated with changing the cutoff of the CFT, at first sight it seems that unitary evolution is not possible. To understand what unitarity means let us consider the CFT on $\mathbb{R} \times S^{d-1}$, which is dual to de Sitter space in static coordinates.

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \,. \tag{6.1}$$

For $r > \ell$ surfaces of constant r are space-like Cauchy surfaces, and the euclidean time t of the CFT is a space-like coordinate. A surface of constant r corresponds to a CFT with a cutoff, for example a lattice with spacing determined by r. As r is decreased the lattice spacing gets bigger, and eventually the whole S^{d-1} is taken up by a single lattice site. This happens when $r = \ell$. At this point the CFT effectively lives on a one dimensional lattice in the t direction, with lattice spacing equal to ℓ.[6]

Now let's regard the CFT from a hamiltonian point of view. At any euclidean time t we can determine the state of the CFT, for example by doing a euclidean path integral, possibly with some operator insertions if the bulk de Sitter space is not in its vacuum state. The de Sitter spacetime corresponds to a whole history of CFT states, one state at each value of euclidean time. The question is what CFT states are allowed to be part of this history.

de Sitter time evolution corresponds to RG transformations in the CFT. We want to be able to coarsen the lattice to the point where $r = \ell$, and still distinguish the different histories of the CFT. But as we saw in section 3.4, a CFT only has approximately c degrees of freedom per lattice site. This puts a limit on the number of histories that we can distinguish. Note that c is also proportional to the de Sitter entropy. Thus de Sitter entropy is a measure of how many states in the CFT can be involved in a description of unitary time evolution in de Sitter space.

These unitarity bounds also appear in the analytic continuation from de Sitter to AdS that we discussed in section 4. To see this, we first recall some properties of the AdS/CFT correspondence. Consider a lorentzian CFT on $\mathbb{R} \times S^{d-1}$. This CFT is dual to global AdS space. AdS states with energy $E > M_{\mathrm{bh}}$, where

$$M_{\mathrm{bh}} = \frac{(d-1)\,\mathrm{vol}(S^{d-1})\,\ell^{d-2}}{8\pi G}\,, \tag{6.2}$$

will form a stable black hole, while states with energy $E < M_{\mathrm{bh}}$ will at most form an unstable black hole that subsequently decays [26, 27]. Stability is determined by comparing the euclidean action of the black hole to the free energy of a thermal gas. In the coordinates we are using, the black hole is stable if the horizon radius r_0 satisfies $r_0 > \ell$. Note that the smallest stable AdS black hole (the one with $r_0 = \ell$) has an entropy $\mathrm{vol}(S^{d-1})\ell^{d-1}/4G$ which is exactly equal to the entropy of de Sitter space. So if one gives the AdS CFT an entropy which exceeds the entropy of the corresponding de Sitter space, a stable black hole forms inside AdS.

[6] To enter the region $r < \ell$ one should keep an appropriate subset of the degrees of freedom on these remaining lattice sites. This is a subtle problem, discussed for example in [24].

What does this mean when we analytically continue to de Sitter space? The metric of the AdS black hole is

$$\begin{aligned} ds^2 &= -\left(1+\frac{\tilde{r}^2}{\ell^2}-\frac{w_d M}{\tilde{r}^{d-2}}\right)d\tilde{t}^2+\left(1+\frac{\tilde{r}^2}{\ell^2}-\frac{w_d M}{\tilde{r}^{d-2}}\right)^{-1}d\tilde{r}^2+\tilde{r}^2 d\Omega_{d-1}^2 \\ w_d &= \frac{16\pi G}{(d-1)\,\mathrm{vol}(\mathrm{S}^{\mathrm{d}-1})} \end{aligned}$$

and the substitution $r = i\tilde{r}$ gives three possibilities.

- For odd d one gets an imaginary mass for the de Sitter black hole.
- For $d = 4n + 2$ one gets a black hole in de Sitter space with a negative mass, which gives a naked singularity.
- For $d = 4n$ one gets a positive mass black hole in de Sitter space. However the mass of a de Sitter black hole is bounded above, $M < M_{\mathrm{max}}$, by the requirement that the black hole horizon is smaller than the cosmological horizon (otherwise a naked singularity appears). One finds that $M_{\mathrm{bh}} > M_{\mathrm{max}}$, so even in this case one gets a naked singularity.

Thus in all cases, a stable black hole in AdS space corresponds to a naked singularity in de Sitter space.

This shows that if we exceed the mass (or entropy) limit in the CFT naked singularities will form in de Sitter space.[7] These singularities can be thought of as symptoms of the breakdown of unitary time evolution. On the CFT side exceeding the entropy bound means the CFT undergoes a phase transition [27]. Thus only the low energy phase of the CFT on $\mathbb{R} \times S^{d-1}$ describes de Sitter space.

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References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[2] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[3] For reviews see O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111];
W. Taylor, *M(atrix) theory: matrix quantum mechanics as a fundamental theory*, *Rev. Mod. Phys.* **73** (2001) 419 [hep-th/0101126].

[7] This has also been argued in [28].

[4] A. Strominger, *The dS/CFT correspondence*, *J. High Energy Phys.* **10** (2001) 034 [hep-th/0106113].

[5] J. Maldacena and A. Strominger, *Statistical entropy of de Sitter space*, *J. High Energy Phys.* **02** (1998) 014 [gr-qc/9801096];
C.M. Hull, *Timelike t-duality, de Sitter space, large-N gauge theories and topological field theory*, *J. High Energy Phys.* **07** (1998) 021 [hep-th/9806146];
M. Bañados, T. Brotz and M.E. Ortiz, *Quantum three-dimensional de Sitter space*, *Phys. Rev.* **D 59** (1999) 046002 [hep-th/9807216];
S. Hawking, J. Maldacena and A. Strominger, *Desitter entropy, quantum entanglement and AdS/CFT*, *J. High Energy Phys.* **05** (2001) 001 [hep-th/0002145];
V. Balasubramanian, P. Hořava and D. Minic, *Deconstructing de Sitter*, *J. High Energy Phys.* **05** (2001) 043 [hep-th/0103171];
E. Witten, *Quantum gravity in de Sitter space*, hep-th/0106109;
M. Li, *Matrix model for de Sitter*, hep-th/0106184;
S. Nojiri and S.D. Odintsov, *Conformal anomaly from dS/CFT correspondence*, *Phys. Lett.* **B 519** (2001) 145 [hep-th/0106191];
E. Silverstein, *(A)dS backgrounds from asymmetric orientifolds*, hep-th/0106209;
D. Klemm, *Some aspects of the de Sitter/CFT correspondence*, *Nucl. Phys.* **B 625** (2002) 295 [hep-th/0106247];
A. Chamblin and N.D. Lambert, *Zero-branes, quantum mechanics and the cosmological constant*, *Phys. Rev.* **D 65** (2002) 066002 [hep-th/0107031];
B. McInnes, *Exploring the similarities of the dS/CFT and AdS/CFT correspondences*, *Nucl. Phys.* **B 627** (2002) 311 [hep-th/0110062];
V. Balasubramanian, J. de Boer and D. Minic, *Mass, entropy and holography in asymptotically de Sitter spaces*, hep-th/0110108;
U.H. Danielsson, *A black hole hologram in de Sitter space*, hep-th/0110265;
R.-G. Cai, *Cardy-verlinde formula and asymptotically de Sitter spaces*, *Phys. Lett.* **B 525** (2002) 331 [hep-th/0111093];
E. Halyo, *On the Cardy-Verlinde formula and the de Sitter/CFT correspondence*, hep-th/0112093; *Strings and the holographic description of asymptotically de sitter spaces*, hep-th/0201174;
S.R. Das, *Thermality in de Sitter and holography*, hep-th/0202008;
F. Leblond, D. Marolf and R.C. Myers, *Tall tales from de Sitter space, 1. Renormalization group flows*, hep-th/0202094;
L. Dyson, J. Lindesay and L. Susskind, *Is there really a de Sitter/CFT duality*, hep-th/0202163.

[6] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics and particle creation*, *Phys. Rev.* **D 15** (1977) 2738.

[7] R. Bousso, *A covariant entropy conjecture*, *J. High Energy Phys.* **07** (1999) 004 [hep-th/9905177]; *Holography in general space-times*, *J. High Energy Phys.* **06** (1999) 028 [hep-th/9906022].

[8] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[9] A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [hep-th/9809022].

[10] L. Bombelli, R.K. Koul, J.-H. Lee and R.D. Sorkin, *A quantum source of entropy for black holes*, *Phys. Rev.* **D 34** (1986) 373.

[11] M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71** (1993) 666 [hep-th/9303048].

[12] See for example R.B. Ash, *Information theory*, Dover, 1990;
T.M. Cover and J.A. Thomas, *Elements of information theory*, Wiley, 1991;
J. Preskill, *Lecture notes on quantum computation*, available at
http://www.theory.caltech.edu/people/preskill/ph229/.

[13] H.W.J. Bloete, J.L. Cardy and M.P. Nightingale, *Conformal invariance, the central charge and universal finite size amplitudes at criticality*, *Phys. Rev. Lett.* **56** (1986) 742;
I. Affleck, *Universal term in the free energy at a critical point and the conformal anomaly*, *Phys. Rev. Lett.* **56** (1986) 746.

[14] V. Balasubramanian, P. Kraus, A.E. Lawrence and S.P. Trivedi, *Holographic probes of anti-de Sitter space-times*, *Phys. Rev.* **D 59** (1999) 104021 [hep-th/9808017].

[15] G. Lifschytz and V. Periwal, *Schwinger-Dyson = Wheeler-DeWitt: gauge theory observables as bulk operators*, *J. High Energy Phys.* **04** (2000) 026 [hep-th/0003179].

[16] C.M. Hull, *de Sitter space in supergravity and m theory*, *J. High Energy Phys.* **11** (2001) 012 [hep-th/0109213].

[17] P. Candelas and D.J. Raine, *General relativistic quantum field theory — an exactly soluble model*, *Phys. Rev.* **D 12** (1975) 965.

[18] R. Bousso, A. Maloney and A. Strominger, *Conformal vacua and entropy in de Sitter space*, hep-th/0112218.

[19] M. Spradlin and A. Volovich, *Vacuum states and the S-matrix in dS/CFT*, hep-th/0112223.

[20] See for example the appendix to D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, *The operator product expansion for Wilson loops and surfaces in the large-N limit*, *Phys. Rev.* **D 59** (1999) 105023 [hep-th/9809188].

[21] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys. (NY)* **144** (1982) 249.

[22] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104].

[23] S.W. Hawking, *The boundary conditions for gauged supergravity*, *Phys. Lett.* **B 126** (1983) 175.

[24] L. Susskind, *Holography in the flat space limit*, hep-th/9901079.

[25] T. Banks, *Cosmological breaking of supersymmetry or little lambda goes back to the future, 2*, hep-th/0007146;
R. Bousso, *Positive vacuum energy and the N-bound*, *J. High Energy Phys.* **11** (2000) 038 [hep-th/0010252].

[26] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577.

[27] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[28] V. Balasubramanian, J. de Boer and D. Minic, *Mass, entropy and holography in asymptotically de Sitter spaces*, hep-th/0110108;
F. Leblond, D. Marolf and R.C. Myers, *Tall tales from de Sitter space, 1. Renormalization group flows*, hep-th/0202094.